Supplementary data for

Laminar-to-Turbulence Transition Revealed Through a Reynolds Number Equivalence

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1. Appendix A: Pipe flow

In this section, γ (ratio of local inertia effect to viscous effect) functions in the fully developed flows (laminar and turbulent, respectively) in a smooth straight pipe are obtained. The system considered is of the cylindrical system (x, r), where x is along the pipe and r is the radial coordinate. The fully developed laminar fluid motion is a one-dimensional case, which is described using the parabolic profile $\bar{u}/u_{\max}$ = 1 - $(r/R)^2$, where u is the fluid velocity in the x-direction, $u_{\max}$ is the maximum velocity, R is the radius of pipe. The maximum fluid velocity occurs at the center $u_{\max} = 2u_{\mathrm{m}}$, where u_{m} is the mean velocity. γ can be obtained as a function of r/R at first for different Reynolds numbers (Re(s)), where Re is defined with u_{m}. Using the relationship between u and r, γ can be further obtained as a function of $\bar{u}/u_{\mathrm{m}}$:

$$\gamma_{xx} = \frac{\rho \bar{u}^2}{\mu\left|\frac{\partial \bar{u}}{\partial y}\right|} = \frac{1}{2}\mathrm{Re}\left[\left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{-\frac{1}{2}} - 2\left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{\frac{1}{2}} + \left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{\frac{3}{2}}\right] \quad \text{(S1)}$$

Where ρ is the fluid density, μ is the fluid viscosity, the subscript xx indicates that the inertia effects in the x-direction interact with the shear effects applied in the x-direction as well.

For fully developed turbulent flow, the velocity profile is approximated with the 1/7th law $\frac{\bar{u}}{u_{\max}} = \left(1-\frac{r}{R}\right)^{\frac{1}{7}}$, where $\bar{u}$ is the time-averaged local velocity, $u_{\max} \approx \frac{5}{4}u_{\mathrm{m}}$ for $10^4 < \mathrm{Re} < 10^5$ [13]. For regions other than the region extremely close to the wall and the region toward the central line, this approximation is appropriate for use. Similarly, the following is obtained:

$$\gamma_{xx} = \frac{\rho \bar{u}^2}{\mu\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}\right|} \approx 4.375\,\mathrm{Re}\left(\frac{\bar{u}}{u_{\max}}\right)^8 = 4.375\,\mathrm{Re}\left(\frac{4}{5}\cdot\frac{\bar{u}}{u_{\mathrm{m}}}\right)^8 \quad \text{(S2)}$$

1.1. Crossover behavior leading to determination of the onset Reynolds number

A comparison between Eq. (S1) and Eq. (S2) is expected to reveal the different behaviors. If this crossover occurs when a Re is on par with what has been found in the past, whether experimentally or theoretically using direct numerical simulation (DNS), there is a significant chance that this actually corresponds to the critical Re ($\mathrm{Re_c}$). Another significant piece of evidence would be provided if the crossover linking to the onset of turbulence was around $\gamma_{xx} = 25$, or at least in the range of $y^+ \approx 11.5 \pm 5$. The two γ_{xx} for laminar flow and for turbulence can cross as a function of r/R. However, this crossover γ_{xx}, which is about 8000, clearly does not represent the regime change (for conciseness, the result is not shown schematically here). As indicated earlier, even in the laminar regime, the γ_{xx} of all values can be found; therefore, γ_{xx} alone cannot determine the $\mathrm{Re_c}$. How γ_{xx} is influenced by velocity determines the onset of turbulence. Therefore, in order to discern the behaviors of the two regimes, the γ_{xx} as influenced by the velocity is revealed by its derivative against velocity.

Based on Eqs. (S1) and (S2), respectively, the following derivatives can be obtained:

$$I = \left|\frac{\partial \gamma_{xx}}{\partial\left(\frac{\bar{u}}{u_{\mathrm{m}}}\right)}\right| = \frac{1}{2}\mathrm{Re}\left[\frac{1}{4}\left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{-\frac{3}{2}} + \frac{1}{2}\left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{-\frac{1}{2}} - \frac{3}{4}\left(1-\frac{\bar{u}}{2u_{\mathrm{m}}}\right)^{\frac{1}{2}}\right] \quad \text{(S3)}$$

and

$$II = \left| \frac{\partial \gamma_{xx}}{\partial \left(\frac{\bar{u}}{u_{\mathrm{m}}} \right)} \right| = 35 \,\mathrm{Re} \left[\frac{4}{5} \cdot \frac{\bar{u}}{u_{\mathrm{m}}} \right]^{7} \quad \text{(S4)}$$

Now u and $\bar{u}$ are taken to be exchangeable. I represents the laminar regime and II represents the turbulent regime. Determination of the crossover that separates the laminar from the turbulent regime can be sought at II - I = 0:

$$II - I = 35 \left(\frac{\bar{u}}{\frac{5}{4} u_{\mathrm{m}}} \right)^{7} - \frac{1}{2} \left[\frac{1}{4} \left(1 - \frac{\bar{u}}{2u_{\mathrm{m}}} \right)^{-\frac{3}{2}} + \frac{1}{2} \left(1 - \frac{\bar{u}}{2u_{\mathrm{m}}} \right)^{-\frac{1}{2}} - \frac{3}{4} \left(1 - \frac{\bar{u}}{2u_{\mathrm{m}}} \right)^{\frac{1}{2}} \right] = 0 \quad \text{(S5)}$$

It can be shown that at point γ_{xx} = 25 (where y^+ = 5), the crossover yields Re = 2082.75, which makes Eq. (S5) balanced. At this point, u/u_m = 0.597. The obtainment of this critical value is demonstrated in Fig. S1.

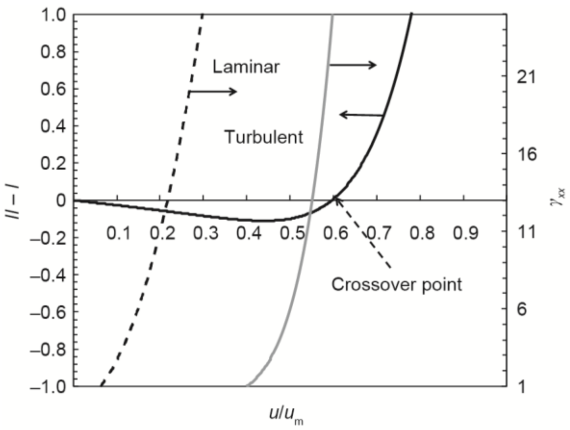

Fig. S1. Obtaining the $\mathrm{Re_c}$ by finding the crossover point corresponding to the laminar-to-turbulence transition in the boundary layer.

It is remarkable that the definition of the local parameter γ_{xx} makes it possible to link the global parameter Re to a local "texture" of the boundary layer (i.e., the velocity distribution described by the Universal Law of the Wall). The sensitivity of the onset Re against the velocity function can be investigated further by employing the following, more generic, formula:

$$\frac{\bar{u}}{u_{\max}} = \left(1 - \frac{r}{R} \right)^{\frac{1}{N}} \quad \text{(S6)}$$

where N is the power of the classic approximation of the time-averaged velocity distribution in turbulence regime, and may be varied to examine the sensitivity, leading to the following:

$$u_{\max} = \frac{N+1}{N} u_{\mathrm{m}}$$

In a good match with classical experimental data [1,2], N = 11 causes the correlation to fall within 1% of the velocity range (not shown here). Nevertheless, the following applies generically:

$$\gamma_{xx} = \frac{\rho \bar{u}^{2}}{\mu \left| \frac{\partial \bar{u}}{\partial y} \right|} = \frac{N+1}{2} \mathrm{Re} \left(\frac{N}{N+1} \cdot \frac{\bar{u}}{u_{\mathrm{m}}} \right)^{N+1} \quad \text{(S7)}$$

The first derivative against velocity can then be similarly obtained:

$$III = \frac{\partial \gamma_{xx}}{\partial \left(\frac{u}{u_{\mathrm{m}}} \right)} = \frac{N(N+1)}{2} \mathrm{Re} \left(\frac{N}{N+1} \cdot \frac{\bar{u}}{u_{\mathrm{m}}} \right)^{N} \quad \text{(S8)}$$

The crossover is sought at III - I = 0:

$$III - I = \frac{N(N+1)}{2}\left(\frac{u}{\frac{N+1}{N}u_{\mathrm{m}}}\right)^{N} - \frac{1}{2}\left[\frac{1}{4}\left(1-\frac{u}{2u_{\mathrm{m}}}\right)^{-\frac{3}{2}} + \frac{1}{2}\left(1-\frac{u}{2u_{\mathrm{m}}}\right)^{-\frac{1}{2}} - \frac{3}{4}\left(1-\frac{u}{2u_{\mathrm{m}}}\right)^{\frac{1}{2}}\right] = 0 \quad \text{(S9)}$$

With $N = 11$, it can be shown that when Re = 2005.75, the velocity at point $y^+ = 5$ ($\gamma_{xx} = 25$) and $u/u_{\mathrm{m}} = 0.650$. This is the $\mathrm{Re_c}$ for the onset (see Fig. S2). The analysis described above indicates that when $\gamma_{xx} = 25$, if the fluid flow at that location has sufficient energy, turbulence occurs.

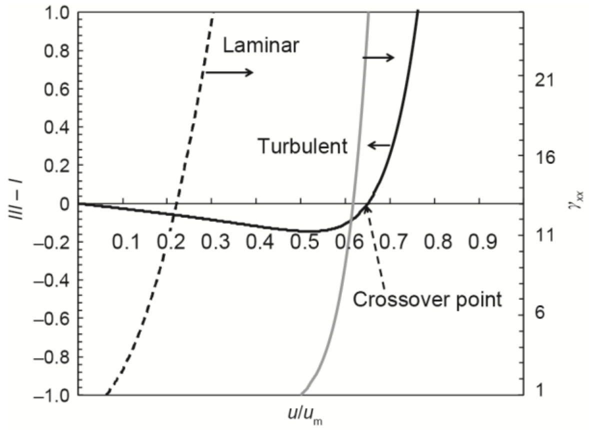

Fig. S2. Obtaining the $\mathrm{Re_c}$ by finding the crossover point corresponding to the laminar-to-turbulence transition in the boundary layer when using Eq. (S7) with $N = 11$.

2. Appendix B: Plate flow

Historical experimental measurements have validated, for a wide range of Re(s), that the similarity solution to the velocity profiles of the plate flow is excellent. It is, however, very useful to correlate the result using a parabolic formula $u/U_\infty = 1-(r/R_{\mathrm{o}})^2$ instead, where U_∞ is the velocity of the bulk fluid in plate flow, R_{o} is the demarcation of boundary layer thickness. $r/R_{\mathrm{o}} = 1-\eta/R_{\mathrm{o}}$, where η is the dimensionless transformation variable, and R_{o} is taken as 3.67, beyond which u is considered to be U_∞ (0.1% error). For any finite x, the following is obtained:

$$\gamma_{xx} = \frac{\rho u^2}{\mu\left|\frac{\partial u}{\partial y}\right|} = \frac{\rho U_\infty^2\left[1-\left(\frac{r}{R_{\mathrm{o}}}\right)^2\right]^2}{-2\mu U_\infty \frac{r}{R_{\mathrm{o}}^2}\cdot\frac{\partial r}{\partial y}} = \frac{R_{\mathrm{o}}}{2^{\frac{1}{2}}}\mathrm{Re}_x^{\frac{1}{2}}\left[\left(1-\frac{u}{U_\infty}\right)^{-\frac{1}{2}} - 2\left(1-\frac{u}{U_\infty}\right)^{\frac{1}{2}} + \left(1-\frac{u}{U_\infty}\right)^{\frac{3}{2}}\right] \quad \text{(S10)}$$

For a turbulent boundary layer in plate flow, it is also possible to have $u/U_\infty = (y/\delta_{\mathrm{t}})^{\frac{1}{7}}$, where δ_{t} is the total thickness of the boundary layer. The total thickness is $\delta_{\mathrm{t}} \approx 0.38x\,\mathrm{Re}_x^{-\frac{1}{5}}$. For any finite x, the following is true:

$$\gamma_{xx} = \frac{\rho\overline{u}^2}{\mu\left|\frac{\partial\overline{u}}{\partial y}\right|} = 0.38\times 7\,\mathrm{Re}_x^{\frac{4}{5}}\left(\frac{u}{U_\infty}\right)^8 \quad \text{(S11)}$$

Again, crossover can happen if Eq. (S12) is directly compared with Eq. (S14); however, as in the case of pipe flow, the crossover γ_{xx} is about 800–900 (for conciseness, the result is not shown schematically here). This is not appropriate for describing transition. The following can then be derived:

$$IV = \left|\frac{\partial\gamma_{xx}}{\partial\left(\frac{u}{U_\infty}\right)}\right| = \frac{R_{\mathrm{o}}}{2^{\frac{1}{2}}}\mathrm{Re}_x^{\frac{1}{2}}\left[\frac{1}{2}\left(1-\frac{u}{U_\infty}\right)^{-\frac{3}{2}} + \left(1-\frac{u}{U_\infty}\right)^{-\frac{1}{2}} - \frac{3}{2}\left(1-\frac{u}{U_\infty}\right)^{\frac{1}{2}}\right] \quad \text{(S12)}$$

and

$$V = \left| \frac{\partial \gamma_{xx}}{\partial \left(\frac{\bar{u}}{U_\infty} \right)} \right| = 0.38 \times 7 \times 8 \, \mathrm{Re}_x^{\frac{4}{5}} \left(\frac{\bar{u}}{U_\infty} \right)^7 \quad \text{(S13)}$$

The solution for $V - IV = 0$ (the crossover) for errors within 0.1% was obtained by setting $\mathrm{Re}_{x,c} = 5.5 \times 10^5$. This is the well-known Re_c for the onset of turbulence in plate flow. This solution yields a corresponding $\gamma_{xx} = 154.6$ ($y^+ = 12.4$), which falls within the correct range for turbulent action to take effect, as mentioned in the introduction. At this point, the calculation shows that $u/U_\infty = 0.441$, according to the 1/7 power law. This situation is illustrated in Fig. S3. In fact, both the existing literature and classroom teaching on turbulence tend to mention a range for the onset for plate flow, such as a $\mathrm{Re}_{x,c}$ that ranges from 10^5 to 10^6. If $\gamma_{xx} = 89.24$ ($y^+ = 9.45$), a $\mathrm{Re}_{x,c}$ of 10^5 would be obtained. At this point, $\bar{u}/U_\infty = 0.487$.

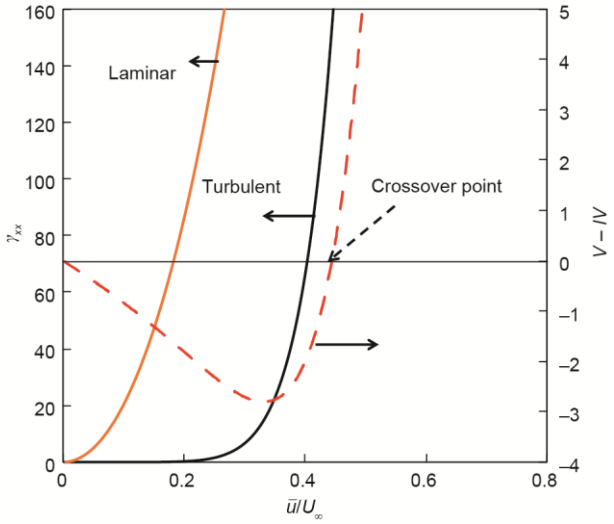

Fig. S3. Obtaining the Re_c by finding the crossover point for plate flow.

References

[1] Nikuradse J. Gesetzmassigkeiten der turbulenten stromung in glatten rohren. Berlin: VDI Verlag; 1932. German.
[2] Pai SI. On turbulent flow in circular pipe. J Franklin Inst 1953;256(4):337–52.